Portraying mathematics teachers' knowledge for teaching the addition of fractions through representations

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Abstract: Teaching the concept of fractions to students stays challenging, yet representations define an effective strategy to overcome such a challenge. Considering the impact of teachers' knowledge on students' achievement, this study aimed at portraying mathematics teachers' knowledge for teaching fractions through representations, precisely, the addition process that remains a prerequisite to other operations. Hence, a purposefully selected sample of novice mathematics teachers was asked to propose a pedagogical activity through which the addition of fractions could be taught to early-age students. Later, their responses were analyzed through the study framework, which was developed by combining the five interrelated constructs of fractions with the types of activities used when teaching the addition of fractions. As a result, teachers' knowledge was crystallized into three principal categories of utilizing the Part-whole, Measure, and Operator constructs. Furthermore, the related concepts of the unit and proportional equivalence, the fractional unit, including the iteration process, and the connection between addition and subtraction were discussed. Also, manners of representing (1) the added fractions and the result through two distinct models, (2) the added fractions and the result jointly in one

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model, and (3) only the added fractions emerged. These results provide a foundation for the professional development of mathematics teachers.

Keywords: Representations, Fractions, Teachers’ knowledge, Mathematics teaching

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INTRODUCTION

On the one hand, fractions consider one of the most challenging areas in school mathematics, either for students to learn or, pedagogically, for teachers to teach (Bruce, Chang, & Flynn, 2013; Copur-Gencturk, 2021; Getenet & Callingham, 2017; Gupta & Wilkerson, 2015; Newton, 2008; Siegler, Thompson, & Schneider, 2011; Vamvakoussi & Vosniadou, 2010). The way fractions are taught through pursuing specific algorithms without paying enough attention to develop students' understanding regards one source for students' encountered difficulties (Siemon et al., 2015). Alternatively stated, how teachers introduce the concept of fractions to students remains significant in developing their understanding. This highlights the widely stated term entitled Pedagogical Content Knowledge (PCK), which was initially defined by Shulman (1987) as a "special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding" (Shulman, 1987, p. 8).

Throughout three decades of continuous research in different content areas, a consensus has been evident regarding the impact of teachers' PCK on students' achievement (e.g., Baumert et al., 2010; Darling-Hammond & Sykes, 2003; Schacter & Thum, 2004; Stigler & Hiebert, 1999). More specifically, about fractions, several studies have declared a similar association between teachers' knowledge and students' performance (e.g., Kutub, Wijayanti, & Manuharawati, 2019; Ma, 1999; Tobias, 2013). In other words, the students might have a limited understanding of fractions due to how their teachers interpret them (Ribeiro & Jakobsen, 2012); thus, it is crucial to clarify teachers' knowledge for teaching fractions.

On the other hand, representations have an essential role in the theory of mathematics teaching and learning (Mainali, 2021), through which students'

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understanding of mathematical concepts could be facilitated; they are defined by Duval (2006) as something that refers to something else. Representations exemplify one of the five process standards stipulated by the National Council of Teachers of Mathematics (NCTM) since all instructional programs should enable students to "create and use representations to organize, record, and communicate mathematical ideas" (NCTM, 2000, p. 67). Furthermore, implementing representations describes an effective teaching practice while “effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving” (NCTM, 2014, p. 24). Accordingly, representations are often sharpened in mathematics education research; they are required to be understood by students and necessary to be practiced by teachers. That is, teachers, themselves, must have a clear insight into how mathematical concepts could be taught through representations (Samsuddin & Retnawati, 2018).

On the relationship between teaching fractions and representations, representations determine an effective way to teach fractions. For instance, Atagi, DeWolf, Stigler, and Johnson (2016) asserted that teachers should start with visual representations before moving to abstract symbolic ones to develop students’ understanding of fractions. Additionally, other studies declared the effectiveness of teaching operations on fractions through representations (e.g., Dey & Dey, 2010; Mendiburo & Hasselbring, 2010; Siegler et al., 2011; Watanabe, 2002). From this perspective, the current study sheds light on mathematics teachers’ knowledge for teaching fractions through representations, more specifically, the addition of fractions that stays an essential prerequisite to the other three operations (i.e., subtraction, multiplication, and division) (Abbas, Shahrill, & Prahmana, 2020).

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As reported in the literature, on one side, research concerning fractions has mostly focused on students' understanding (e.g., Dhlamini & Kibirige, 2014; Gabriel et al., 2013; Gunawan, Putri, & Zulkardi, 2017), while limited studies highlighted teachers' knowledge (Copur-Gencturk, 2021; Ribeiro & Jakobsen, 2012). On the other side, although multiple studies have sharpened issues of teaching fractions, they primarily concentrated on the division operation (Ma, 1999; Newton, 2008; Olanoff, Lo, & Tobias, 2014). Thus, this study aimed to portray mathematics teachers' knowledge for teaching the addition of fractions through representations, by which the lack of research in this area could be filled, and approaches for professional development could be proposed.

THEORETICAL PERSPECTIVE AND ANALYTICAL FRAMEWORK

Theoretical Perspective

- *Mathematics teachers' knowledge*

Shulman's (1987) research, particularly the notion of PCK, stays essential for any work related to teachers' knowledge. Building on Shulman's idea, several researchers attempted to define the components of PCK in different areas of study; Ball, Thames, and Phelps's (2008) framework constitutes a prominent clarification of PCK for mathematics teachers; namely, Mathematical Knowledge for Teaching (MKT). MKT specifies "the mathematical knowledge that teachers use in the classroom to produce instruction and student growth" (Hill, Ball, & Schilling, 2008, p. 374). It has two principal categories of Subject Matter Knowledge and PCK that incorporates Knowledge of Content and Students, Knowledge of Content and Curriculum, and Knowledge of Content and Teaching (KCT) targeted in this study.

http://dx.doi.org/10.29009/ijres.5.4.8

KCT combines knowing about teaching and knowing about the content taught (i.e., teaching the addition of fractions to students through representations). In other words, it requires an interaction between understanding the mathematics itself (i.e., understanding fractions) and pedagogical issues that affect students' learning (e.g., determining the appropriate representation, selecting the suitable task to consider) (Ball et al., 2008; Hill et al., 2008; Jing-Jing, 2014; Petrou & Goulding, 2011). This matches the original ideas of Ball et al. (2008), as they considered teachers' ability to "evaluate the instructional advantages and disadvantages of representations used to teach a specific idea and identify what different methods and procedures afford instructionally" (p. 401) part of their KCT. It is also consistent with López-Martín, Aguayo-Arriagada, and García López's (2022) description of KCT in the case of fractions as knowing "the different ways of representing a fraction (discrete models and continuous models)" (p.3).

- *Teaching fractions through representations*

Fractions define a bipartite structure of numerator and denominator; they are the only number "that simultaneously represents a magnitude and a division relationship between the numerator and denominator" (Atagi et al., 2016, p. 2). The notion of fractions stays fundamental for students to acquire, not only because of its role in learning other related complex concepts (e.g., proportional, spatial, and algebraic reasoning; rational numbers; probability) but also for practicing daily-life activities (Bruce & Ross, 2009; Gabriel et al., 2013; Siegler et al., 2011; Van Steenbrugge, Lesage, Valcke, & Desoete, 2014). Nonetheless, pedagogically, challenges of how fractions should be taught to students in the mathematics classroom still exist (Getenet & Callingham, 2017; Siemon et al., 2015).

As reported in Newton's (2008) study, there is evidence that pre-service

http://dx.doi.org/10.29009/ijres.5.4.8

and in-service teachers do not necessarily have the knowledge required for teaching fractions; moreover, teachers might have similar misconceptions shared among their students (Ma, 1999; Tirosh, 2000; Zhou, Peverly, & Xin, 2006). Accordingly, research on teachers' knowledge for teaching fractions has been extensively articulated, wherein the influence of teachers' knowledge on students' achievement remains the authentic motive for such research. For example, Tobias (2013) noted that when teachers utilize inaccurate language while teaching, students might continue to practice incorrect language to define fractions. This matches findings of research on pre-service teachers since Van Steenbrugge et al. (2014) outlined their limited conceptual and procedural knowledge required for teaching fractions and, recently, López-Martín et al. (2022) explained their errors related to the meaning of fractions as operators, which might prevent them from maintaining the adequate pedagogical knowledge to teach it to their future students.

Among approaches employed to teach fractions effectively, representations are highlighted, through which students could gain a deeper understanding of this concept (Widodo & Ikhwanudin, 2020). This is particularly emphasized during the early grades, wherein fractions are introduced to students through visual representation (Atagi et al., 2016). Representations describe something that refers to something else (Duval, 2006); and express how the ideas are constructed in individuals' minds (Janvier, 1987). They portray "objects, physical properties, actions and relationships, or objects that are much more abstract" (Goldin, 1998, p. 4, as cited in Godino & Font, 2010). Accordingly, such representations are classified in the literature as (A) external (graphic, pictures, equations, tables) vs. internal (i.e., mental schemes); (B) concrete, representational, and abstract; or (C) enactive, iconic, and symbolic (Mohamed, Ghazali, & Samsudin, 2021).

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Generally, the significance of teaching mathematical concepts through representations has been widely reported. It helps learners express their ways of thinking (i.e., mental models) and overcome the expected difficulties they might face when learning these concepts (Samsuddin & Retnawati, 2018). Hence, representations work as tools through which several abstract notions can be visualized (DeWolf, Grounds, Bassok, & Holyoak, 2014; Mainali, 2021), and fractions symbolize a mathematical concept that can be concretized through these representations.

Reys et al. (2012, as cited in Getenet & Callingham, 2017) explained how students could be encouraged to recognize fractions by representing them on number lines after working on strips folding. That is consistent with Riccomini's (2011, as cited in Widodo & Ikhwanudin, 2020) argumentation regarding the usefulness of teaching fractions through number lines on students' performance. Such effectiveness did not only emerge when working with young students but also with adults, as Atagi et al. (2016) concluded that college students performed significantly better when accurate visuals were provided. More specifically, concerning operations on fractions, the study of Cramer, Wyberg, and Leavitt (2008) demonstrated the usefulness of representations in supporting students' understanding of algorithms for adding fractions. Also, Brijlall (2014) recommended operating concrete objects to fit students' different learning styles and help them overcome difficulties while adding fractions. Similarly, with the usage of fraction bars and number lines, the students were able to solve activities of addition of fractions (either with the same or different denominators) and develop their conceptual understanding of fractions (Gunawan et al., 2017).

According to the Egyptian national curriculum, teaching fractions to students through representation considers a fundamental teaching skill for

http://dx.doi.org/10.29009/ijres.5.4.8

mathematics teachers, particularly in early grades. The concept of fractions (part-whole) is first introduced to students through representations in grade 1 within the geometry content. As reported, students should "partition circles and/or rectangles into two and four equal shares, describe the shares using the words halves, fourths, and quarters; and describe the whole as two of or four of the shares" (Ministry of Education and Technical Education [MOETE], Mathematics Teachers' guide, grade 1, term 2, 2018, p. 7). Moreover, at the end of grade 3, fractions are formally instructed to students, wherein they are required to:

a. Describe a proper fraction 1/b as the quantity formed by 1 part when a whole is partitioned into b equal parts; hence, determining that the proper fraction b/b equals one whole.

b. Identify and represent fractions on a number line.

c. Identify and generate simple equivalent fractions, and explain when and why two fractions are equivalent verbally or by utilizing fraction models.

d. Demonstrate understanding that comparisons of fractions are valid only if the wholes are the same; accordingly, compare two fractions by reasoning about their size using a number line or concrete models.

e. Use concrete models to add and subtract fractions.

(MOETE, Mathematics Teachers' guide, 2020, grade 3, term 2, p. 11.).

This is compatible with global standards. For example, among NCTM's expectations, students in prekindergarten through grade 2 should "understand and represent commonly used fractions, such as 1/4, 1/3, and 1/2" (NCTM, 2000, p. 78); furthermore, students in grades 3 to 5 should "develop understanding of fractions as parts of unit wholes, as parts of a collection, as

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locations on number lines, and as divisions of whole numbers" (ibid., p. 148).

The Study Framework

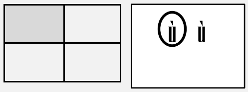
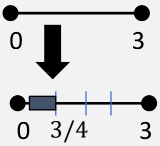
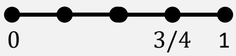
The framework employed in this study depended on two primary ideas: (1) the five interrelated constructs of fractions (Behr, Lesh, Post, & Silver, 1983; Bruce et al., 2013) and (2) the types of pedagogical activities implemented while teaching the addition of fractions (Brijlall, 2014). These ideas are first clarified as follows:

- *The five interrelated constructs of fractions (Multifaceted nature of fractions)*

Indeed, the concept of fractions has multiple interpretations; it can be conceptualized into five constructs: part-whole, ratio, operator, quotient, and measure (Behr et al., 1983); Table 1 defines their meaning and gives a possible example for each interpretation (Bruce et al., 2013; Dhlamini & Kibirige, 2014; Getenet & Callingham, 2017). As reported in Bruce et al.'s (2013) study, an emphasis should be placed on such various interpretations of fractions during formal instruction due to their effectiveness in promoting students' conceptual understanding. Considering this, insights into the diversity of fractions' interpretations involved during teaching the addition of fractions through representations could be provided. Alternatively stated, since each construct of fractions has a related model or a relevant representation (see Table 1), thus, through this representation, the utilized construct by teachers could be deduced.

http://dx.doi.org/10.29009/ijres.5.4.8

Table 1. The Five Interrelated Constructs of Fractions

Construct	Meaning	Example	Related models
Part-whole	It occurs when a whole object is partitioned into parts of equal size (i.e., the relationship between the whole and its partitions).	When a paper sheet is divided into four equal pieces, 1/4 defines one partition among these four pieces.	Continuous (e.g., area or a volume) or discrete models 
Ratio	It expresses a relationship (or comparison) between two quantities (i.e., part-part relationship).	When we have a group of three cats and four gods, 3/4 indicates the ratio of cats to dogs.	linear, continuous, or discrete models.
Operator	It is indicative of a multiplication (or division) process and is related to functions applied consecutively.	3/4 defines 3 divided by 4, or 3 multiplied by 1/4.	
Quotient	It defines fractions as the whole number resulting from dividing the numerator by the denominator.	3/4 represents the result of the division process, which is 3 ÷ 4.	Quotient model 
Measure	It refers to a number ordered on a number line based on its distance from zero (linear interpretation).	3/4 represents a certain point on the number line.	Number line (length) models 

- *Types of pedagogical activities implemented while teaching the addition of fractions*

In this study, participated teachers were unrestricted about the required activity in which they were encouraged to think of suitable scenarios upon their viewpoints while teaching the addition of fractions (see the Method section). This was intended by the researcher to widen the diversity of teachers' provided

http://dx.doi.org/10.29009/ijres.5.4.8

activities, hence, the relevant representations. For this purpose, Brijlall's (2014) determination of types of implemented activities, progressed in terms of difficulty, that are usually involved while teaching the addition of fractions in the mathematics classroom was reasonable to utilize. These types are the addition of fractions with the same denominators, different denominators when one denominator is a multiple of the other, different denominators when none of the denominators is a multiple of the other, and the addition of mixed fractions. Such a sequence of activities is typically aligned with the progress of the national school mathematics textbooks. However, as mentioned before, this does not mean that teachers were asked to illustrate each type; they were, rather, free to create their preferred activity.

Through synthesizing the above-clarified ideas, the study framework has been concretized in Table 2 as follows:

Table 2. The Study Framework

The Activity type	**The Utilized Construct**				
	Part-whole	Ratio	Operator	Quotient	Measure
Type 1: Addition of fractions with the same denominators.					
Type 2: Addition of fractions with different denominators when one denominator is a multiple of the other.					
Type 3: Addition of fractions with different denominators when none of the denominators is a multiple of the other.					
Type 4: Addition of mixed fractions.					

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METHOD

Participants and context of the study

The study sample consisted of twelve mathematics teachers who joined a professional development diploma at the Faculty of Education, Tanta University, Egypt, in the academic year 2021/2022. They were selected purposely considering their highest academic degree and teaching experience. The participants all recently graduated from either the Faculty of Engineering or the Faculty of Science (with no educational background) and have a novice teaching experience between one to three years (at most).

As part of this diploma, teachers should join the micro-teaching course, which aims at discussing topics of school mathematics, especially in the early grades. Considering this, the participants were asked, upon their experience, to select a mathematical concept that is difficult for young students to acquire; consequently, fractions were one among the raised concepts. All teachers who joined the micro-teaching course (sixteen teachers) agreed that fractions remain problematic for students to learn, and in some cases, it is further a challenge to teach. Accordingly, the researcher, who was the instructor for this course, asked them to review the school textbooks (or other sources they have) to see how fractions (and operations on fractions) could be facilitated to students, which will be the topic of the next session.

During this time, the study participants (twelve teachers) were selected upon the criteria described above, wherein four teachers were excluded because of their either exceptional expertise in teaching mathematics for early grades (2 cases) or reluctance to partake in the study (2 cases). One week later, the session was warmed up through some introductory questions discussed between the researcher and participants as follows:

http://dx.doi.org/10.29009/ijres.5.4.8

- Q1 (The researcher): When are fractions first introduced to students according to the national curriculum?
- A1 (The participants): Fractions are first presented to students in grade 1 when they learn 2-D geometrical shapes.
- Q2 (The researcher): Why do you think learning fractions is necessary for students to acquire in this early stage?
- A2 (The participants): Fractions stay valuable for students to practice several daily activities; for example, it is used to share equal quantities of something (e.g., food, money) among different people. Moreover, the concept of fractions is essential while learning other topics, such as determining the time that demands reading the clock (e.g., quarter hour, half hour).
- Q3 (The researcher): What operations could be done on fractions?
- A3 (The participants): Like whole numbers, students can add, subtract, multiply, and divide fractions.

At this stage, the researcher indicated that the addition of fractions, which remains essential for the other three operations, will be sharpened. Also, the participants were informed of the study's purpose, and their consent to share answers and responses (anonymously) was obtained. Accordingly, they were requested to think of the following question and write down their answers on individual sheets.

The question asked to participants

As you all know, at the primary stage, students learn the concept of fractions and, further, conduct operations on fractions; try to reflect on your mathematical knowledge and experience to portray how could the process of addition of fractions be taught to those early age students through

http://dx.doi.org/10.29009/ijres.5.4.8

representations?

In this question, knowledge of teaching and content does an amalgam, which defines teachers' KCT. That is, KCT, in this study, outlines teachers' ways of representations while teaching the addition of fractions (Ball et al., 2008; López-Martín et al., 2022), and representations indicate visual models such as circles, squares, triangles, number lines, etc.

Then, all participants were given a time of one hour to work on this question; furthermore, they were informed that there is neither an optimal solution (nor a specific number of activities they must propose) in which answers might differ upon their experiences or point of views. This corresponds Copur-Gencturk's (2021) perspective on allowing teachers to utilize any representation they wish wherein their understandings and struggles of the mathematical concepts underpin these representations could be captured. Accordingly, teachers' sheets were collected and prepared to be analyzed through the study framework.

Data analysis and trustworthiness

After collecting teachers' sheets, the researcher firstly tried to classify them considering the study framework. The analysis process included determining types of proposed activity (horizontally) and utilized constructs (vertically). For example, Case 1 in Figure 1 (see the Results) was classified as Type 2 (because the proposed fractions were of two different denominators, one is a multiple of the other) and, at the same time, part-whole construct (as it focused on partitioning the rectangular area models equally). Three weeks later, teachers' activities were analyzed again, and the consistency between both analyses' outcomes (i.e., the categorization process) was scrutinized; this defines the *intracoder reliability*. Additionally, the results were also verified by another researcher who got her master's degree in a topic of teachers' knowledge of fractions to ensure *intercoder reliability*. While *intracoder*

http://dx.doi.org/10.29009/ijres.5.4.8

reliability involves the coder's consistency across time, *intercoder reliability* defines consistency across coders, both of which help ensure the framework's ability to result in the consistent categorization of content (Lacy, Watson, Riffe, & Lovejoy, 2015). Thus, Cohen's kappa coefficients were calculated wherein its values were 0.835 and 0.943 for *interacoder* and *intercoder reliability*, respectively. This indicates excellent agreement between coders (McHugh, 2012), which validates the analytical framework, consequently, the study results.

RESULTS AND DISCUSSION

Based on the study framework, the mathematics teachers' knowledge for teaching the addition of fractions through representation were classified as follows:

Table 3. Classification of Teachers' Knowledge for Teaching the Addition of Fractions through Representations

The Activity Proposed	The Construct Used					Total
	Part-whole	Ratio	Operator	Quotient	Measure	
Type 1: Two fractions with the same denominators.	*3 activities* (2 continuous area models and 1 discrete model)				*2 activities* (Number line models)	***5***
Type 2: Two fractions with different denominators; one denominator is a multiple of the other.	*7 activities* (Continuous area models)					***7***
Type 3: Two fractions with different denominators; none of the denominators is a multiple of the other.	*5 activities* (4 continuous area models and 1 discrete model)					***5***
Type 4: Two mixed fractions.			*1 activity* (Continuous area model)		*1 activity* (Number line model)	***2***
Total	***15***	***0***	***1***	***0***	***3***	***19***

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According to Table 3, nineteen activities were proposed as some teachers presented more than one activity. Moreover, these activities were classified into six categories commonly ordered: (Type 2, Part-whole), (Type 3, Part-whole), (Type 1, Part-whole), (Type 1, Measure), (Type 4, Measure), and (Type 4, Operator), which are next discussed focusing on the representation used to illustrate the process of addition of fractions (vertical columns of Table 3).

- **Utilizing the Part-whole construct to teach the addition of fractions**

As presented in Table 3, fifteen activities were proposed to teach the addition of fractions through the part-whole construct. These activities progressed in terms of difficulty starting from Type 1 (3 activities), passing by Type 2 (7 activities), and reaching Type 3 (5 activities). The favor of utilizing the part-whole interpretation was not surprised; it has been reported in multiple studies (e.g., Getenet & Callingham, 2017; Fuchs et al., 2016, as cited in Mohamed et al., 2021), while the reason for so might be relevant to teachers' knowledge of students' intuitive experience of fractions through fair sharing process (Siemon et al., 2015).

About Type 2 (the most common), most mathematics teachers preferred utilizing continuous area models to teach how two fractions with two different denominators, one denominator is a multiple of the other, could be added. Two common characteristics were observed regarding the proposed activities; teachers mostly preferred (1) *unit fractions whose denominator is either a multiple of 2 or 3* and (2) *models of either rectangles or circles to represent such fractions*. Additionally, the circular models were primarily utilized to illustrate the addition of unit fractions whose denominator is a multiple of 2 (cases 3, 5, 6, and 7), while the rectangular ones were operated when the denominator is a multiple of 3 (cases 1 and 2). Favoring of such models might be provoked by the school curriculum, which strengthens using circles or

http://dx.doi.org/10.29009/ijres.5.4.8

rectangles to portray fractions of halves or quarters. Furthermore, the way teachers construct models to illustrate the process of addition of fractions changed in three manners of representing (1) *the added fractions and the result of the process through two distinct models* (Figure 1), (2) *the added fractions and the result combined into one model* (Figure 2), or (3) *the added fractions only* (Figure 3).

When teachers represented each fraction and the result of their addition (the first manner), the transition between both sides was not clear. Although Case 1 represented 1/3 and 1/6 correctly through rectangular area models, how these fractions were combined into one model of six equal partitions was not evident. On the other side, Case 2 was much better in terms of KCT since the teacher first converted 1/3 to 2/6 to be possibly added to 1/6; accordingly, the result was 3/6, which represents 3 partitions (2 of the first fraction and 1 of the second one) from 6 equal ones. This way, *the concept of equivalence* appeared since 1/3 and 2/6 determined two equivalent fractions with different numerators and denominators but are of the same value. However, Case 2's representations of these equivalent fractions were not identical in terms of areas (see Case 2 in Figure 1); this, specifically, indicates the *proportional equivalence concept* (Pedersen & Bjerre, 2021). In that sense, teachers should be aware of what appropriate representation to explain both *unit equivalence* and *proportional equivalence* to students.

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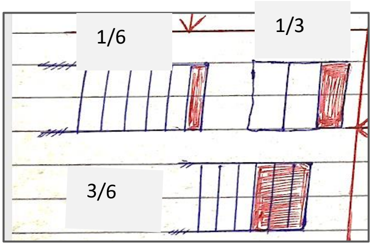

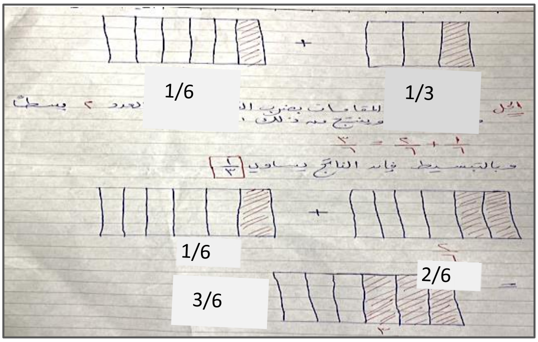

Case 1 Case 2

Figure 1. Sample A responses of representing the addition process through the Part-whole construct

Figure 2 portrays the second manner of teachers' representations, who represented both added fractions and the result in one model. Case 3 modeled 1/2 and 1/4 by half (shaded in red color) and quarter (shaded in green) of the whole circular area; thus, combining both shaded areas specify 3/4. On the other side, Case 4 described the whole as a box that could be divided into 8 small congruent triangles (see Figure 2). Accordingly, 1/2 and 1/8 of this whole box displayed the base (composed of 4 triangles) and 1 more triangle. In this case, the answer would equal 5 triangles; mathematically speaking, 5 of 1/8 = 5/8. *One advantage of operating this model is that it helps students comprehend concepts of the whole and the proper fraction that determines parts of a whole object. Nonetheless, it is not suitable to generalize the process of addition of fractions, particularly the improper or mixed fractions*. This matches Doyle, Dias, Kennis, Czarnocha, and Baker's (2015) argument regarding the Part-whole construct that does not easily demonstrate the concept of an improper fraction.

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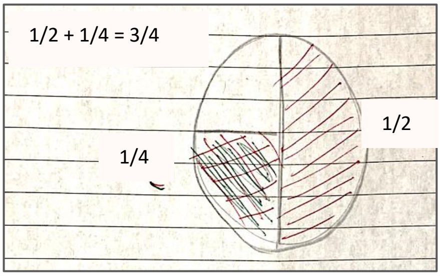

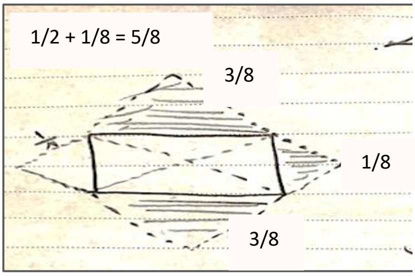

Case 3 Case 4

Figure 2. Sample B responses for representing the addition process through the Part-whole construct

The third manner describes teachers who represented the added fractions only without paying attention to the resultant representative (see Figure 3). Cases 5 and 6 were a bit similar in terms of the selected fractions and the process of teaching how such fractions could be added; nonetheless, they altered in portraying the concept of equivalency. While Case 5 used one model to define 3/4 and its equivalent fraction 6/8 by dividing each 1/4 into two equal partitions (see the red lines used in Case 5 representation), Case 6 represented the equivalent fractions through two separated models. On the other hand, Case 7 sharpened only one fraction, which could be divided in some way to equal the other fraction in the denominator. As the teacher illustrated: the fraction 1/2 could be split into 2 of 1/4 (the two circular models in Case 7 of Figure 3); thus, 1/2 that equals 2/4 could be added to 1/4 since they have the same number of partitions, and the result would be 3/4. In this case, *the equivalence concept was defined through the operator construct; that is, A/B = A/B x N/N = N x (A/NB) while N is a whole number*.

Basically, the three teachers aimed at getting the same number of partitions so that the addition process could be correctly operated, and this was performed by *maintaining the concept of equivalence*. However, Case 5's representation appeared more suitable to young students because portraying the

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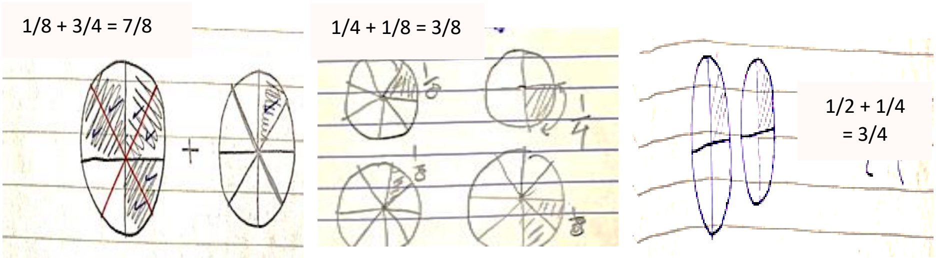
fraction and its equivalent in one model enhances understanding them as one entity (i.e., *unit equivalence*). In other words, if we divided a specific unit into 2, 3, or N partitions, this original unit would not change and would equal the summation of those partitions; mathematically stated, A/B = *N x (A/NB)*. This is consistent with what Getenet and Callingham's (2017) reported, "the ratio, operator, and quotient concepts of fraction were often reflected in part-whole contexts during the dialogue between the teacher and students" (p. 284).



Case 5 Case 6 Case 7

Figure 3. Sample C responses for representing the addition process through the Part-whole construct

After the (Type 2, Part-whole) category, (Type 3, Part-whole) stands in the second rank in terms of popularity among participants. It indicates teachers who employed area models while teaching the addition of fractions with two different denominators when none of the denominators is a multiple of the other (Figures 4 and 5). As observed from the collected data, teachers constantly suggested *one denominator, at least, to be a prime number*. This helped them avoid maintaining a common factor between denominators or converting the result to its simplest form. Again, in this category, manners 1 (i.e., representing both added fractions and the result through two distinct models) and 3 (i.e., representing the added fractions only), which were described before, emerged. Both are further detailed as follows:

As shown in Figure 4, only Case 8 was interested in displaying each

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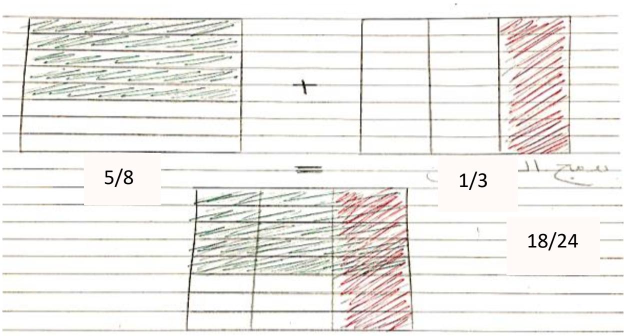
added fraction and the result through three continuous area models. Although KCT of Case 8 seemed more sophisticated than others of the same category, the teacher had a misunderstanding regarding which area represents the result. Case 8 first portrayed 1/3 through a rectangular model divided vertically into three equal partitions, while horizontal lines were used to partition another rectangle into eight equal partitions and get 5/8. Then, both rectangles were combined to calculate 1/3 + 5/8. Until this stage, there were no mistakes; however, when the teacher counted the number of partitions that determined the result, he regarded the crossed area only once. Mathematically speaking, the result had to be 15/24 (came from the green area) + 8/24 (from the red); alternatively, fifteen green partitions + eight red partitions from the whole, which equals 23/24, not 18/24 (as the teacher thought). Perhaps one significant practice in similar situations is to request teachers simply calculate 1/3 + 5/8 and correspond the result to the represented model to reflect on what confusion was done.



Case 8

Figure 4. Sample D response for representing the addition process through the Part-whole construct

Again, manner 3 of representation while performing the addition process appeared in Cases of Figure 5. Nevertheless, Case 12 utilized a discrete area model instead of the continuous models displayed by Cases 9, 10, and 11. As reported before, teachers mostly thought of converting each fraction to an equivalent one to perform the addition process; yet, models of equivalence seemed apparent in Cases 9, 10, and 11 compared to 12. *This resulted because of the explicit representations of the redividing process of these models wherein the original fraction and its equivalent appeared in one model, which resembled case 5's representation of unit equivalence*.

Case 9 attempted to add 1/3 to 1/5 by partitioning 1/3 of the upper rectangular area into 5 equal partitions, while the 1/5 of the lower rectangle was redivided into 3 equal partitions to get the same number of partitions in both models: hence, 1/3 + 1/5 equivalent 5/15 + 3/15, respectively. Case 10 adopted similar procedures while counting 1/2 + 2/3. The equivalent fractions 1/2 and 3/6 were explicitly displayed by the right rectangle, and 2/3 and 4/6 were also portrayed together by the left one; thus, 1/2 + 2/3 would, instead, equal 3/6 + 4/6. Furthermore, Case 11 agreed with Case 10, not only in their procedural knowledge but also concerning the intention of the partitioning technique, which distinguished them from Case 9. As shown in Figure 5, Case 9 redivided merely the parts that would be added together; on the other hand, Cases 10 and 11 repartitioned the whole parts of the original model. This might reflect Pedersen and Bjerre's (2021) *distinction between unit and proportional equivalence*. While unit equivalence defines fractions as equivalent when they have the same parts of equal wholes, proportional equivalence interprets equivalent fractions as the same proportionality across different representations. In other words, *unit equivalence highlights equal parts of congruent models* (Cases 10 and 11), while *proportional equivalence strengthens comparable ratios* (Case 9).

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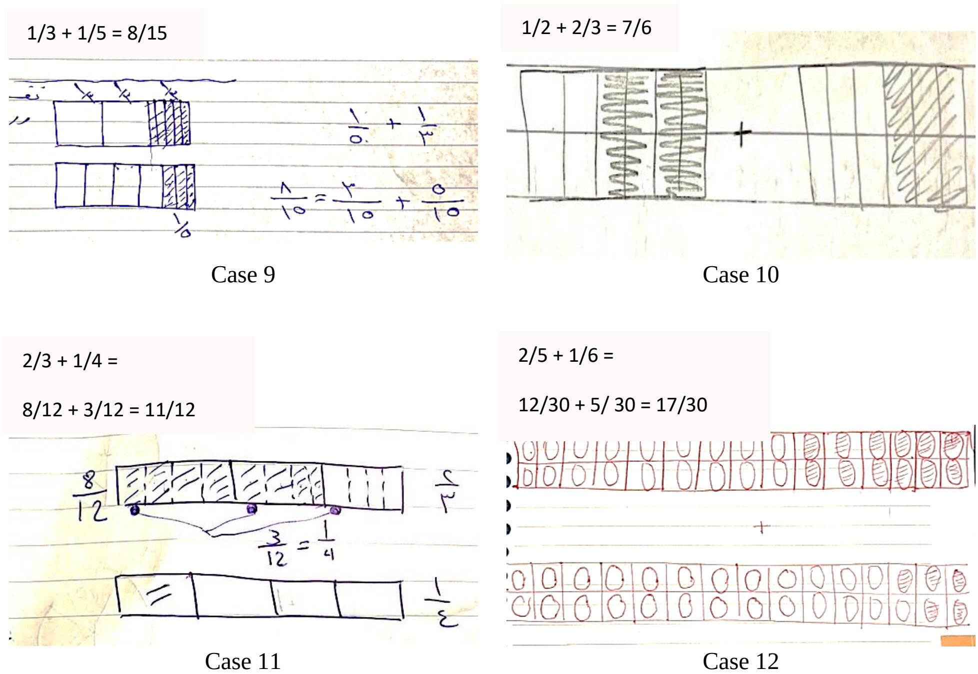

Figure 5. Sample E responses for representing the addition process through the Part-whole construct

The last category of teachers who utilized the Part-whole construct to add fractions is (Type 1, Part-whole), which includes three activities (see Figure 6). Also, and as widespread in this study's results, portraying the added fractions and the results were achieved through continuous area models (Cases 14 and 15), while Case 13 represented 1/5 + 2/5 by combining both continuous and discrete models. Unexpectedly, manner 2 of representation did not appear; rather teacher preferred either manner 1 as in Cases 14 and 15 or manner 3 as in Case 13. As shown in Figure 6, Cases 14 and 15 aimed at figuring out the concept of *the whole* that would be more obvious if they used one model to display the added fraction and the result together (manner 2) instead of separating them. In other words, *manner 2 of representation stayed meaningful if the result would be lower than* (i.e., proper fraction) (or equal) 1.

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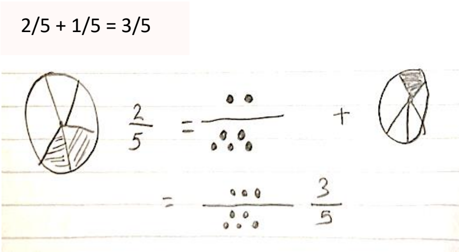
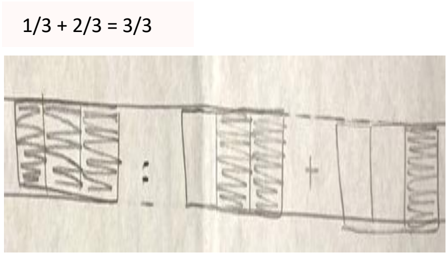
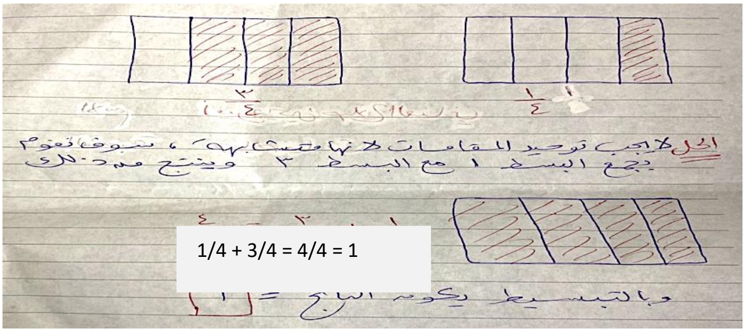
Additionally, although *the whole concept* emerged in this category through which teachers could help their students perceive the relationship between fractions and whole numbers, the notion of equivalence did not appear since there was no need to transform any fraction into an equivalent when fractions of the same denominator.



Case 13



Case 14



Case 15

Figure 6. Sample F responses for representing the addition process through the Part-whole construct

- ***Utilizing the Measure construct to teach the addition of fractions***

Besides the fifteen activities through which the study participants employed the Part-whole construct to teach how to add fractions, the Measure construct emerged in three more activities (see Table 3); two were classified as (Type 1, Measure) (Cases 16 and 17), and another belonged to (Type 4, Measure) (Case 18). In these activities, the added fractions and the results were interpreted as numbers ordered (or a distance from zero) in a number line that exemplifies fractions as quantities in the measure interpretation (Doyle et al.,

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2015; Wong & Evans, 2008). Thus, it was reasonable that (1) *only manner 2 of representations appear* and (2) *improper fractions were proposed* (Cases 16 and 18 in Figure 7) since it does not make sense to express improper fractions through the Part-whole construct wherein the number of parts could not be more than the whole (Stafylidou & Vosniadou, 2004).

Surprisingly, although all teachers were asked to teach the addition of fractions through representations (see the Method section), two teachers instead preferred to model the subtraction process through the Measure construct (Cases 17 and 18 in Figure 7). They further justified this by stating that "addition and subtraction are of two sides of the same coin (subtraction defines negative addition) and utilizing number line helps move around both operations easily." This mirrors Getenet and Callingham's (2017) argument regarding the importance of the Measure construct to add and subtract fractions, particularly fractions of different denominators, wherein they would be interpreted as distances from zero on the same scale.

As noted earlier, Cases 16 and 17 proposed two fractions of the same denominators (Type 1); however, they differed in what numbers should be located on the number line (positive or negative numbers). Case 16 located the origin (i.e., the zero point) in the middle, and the number to the right of this origin are positive while the numbers to the left are negative. This reflects the teacher's *inability to connect between the proposed fractions and the selection of origin's place on a number line*. On the contrary, Case 17 was aware of such a relation; thus, although the teacher preferred a subtraction activity, he located the origin on the left side since the result of the subtraction process would be a positive number. Concretely, Case 16 represented zero as a middle point on the number line; 1, 2, and 3 were on the right; and -1, -2, and -3 were on the left. Additionally, she divided the distance from 1 to 2 into halves (the fractional

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unit) and redefined the positive side of the number line to 0, 1/2, 1, 3/2, and 2 that equals 4/2. Consequently, 3/2 + 1/2 was calculated by locating 3/2 at first, then moving forward one fractional unit (i.e., half) to get 2 or 4/2. Although Case 16 executed the process correctly, neither *the fractional unit concept* was evident, nor the arrows were sketched on the number line. Conversely, the unit fraction was accurately explicit in Case 17's representation, wherein numbers from zero to 6 were divided by 4 (denominators of the proposed fractions). Thus, to calculate 3/4 - 1/4, the teacher started by locating 3/4 and then moved backward one fractional unit (i.e., quarter); hence, the answer would be 2/4. In this case, the arrows were shown through the number line model; this indicated the teacher's understanding of the added fraction and the process as distances from zero on that number line.

Alike Case 16, Case 18 sketched the number line with positive and negative numbers; regardless, the fractional unit concept was absent since the distance between every two whole numbers was not divided into the same units. As shown in Figure 7, to model the subtraction process of 5/4 - 3/5, Case 18 started by converting both fractions to decimals of 1.25 and .60. After, the distance from the origin to -1 (and to 1) was divided into five equal parts to locate -.60 (and the parallel positive number +1). But, when the teacher considered representing 1.25, she divided the distance from 1 to 2 into four equal parts so that the decimal .25 could appear. Thus, 5/4 - 3/5 was defined through two red bars of lengths 1.25 on the right side and .60 on the left; then, the result would be the length of the blue bar, which equals .40 + .25 = .65.

The above argument stresses that teachers' understanding of the fractional unit concept, which is “derived when the standard object or unit of measure is subdivided into smaller equal parts” (Wong & Evans, 2008, p. 579), stays crucial to teaching the addition of fractions through the number line

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model. In that sense, A/B indicates that A measures the unit fraction 1/B, wherein the iteration process is essential to understand that the whole is made up of the repetition of a unit fraction (Wilkins & Norton, 2018); therefore, the Part-whole construct could be extended through the Measure construct to include improper fractions (Doyle et al., 2015).

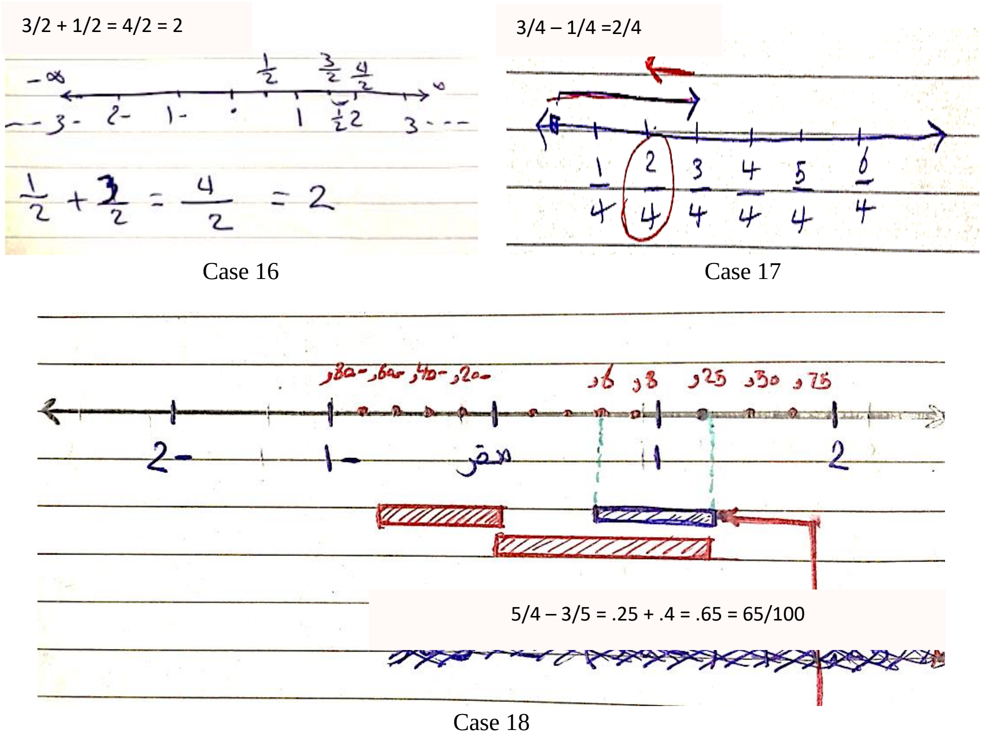

Figure 7. Sample G responses for representing the addition process through the Measure construct

- **Utilizing the Operator construct to teach the addition of fractions**

The last deduced category of teachers' representations was (Type 4, Operator) since the teacher employed the Operator construct to add two mixed fractions (see Figure 8). According to Doyle et al. (2015), the Operator construct expresses an input-output box wherein the output is a fractional amount of the input quantity. Considering this, Case 19 supposed inputs of (I)

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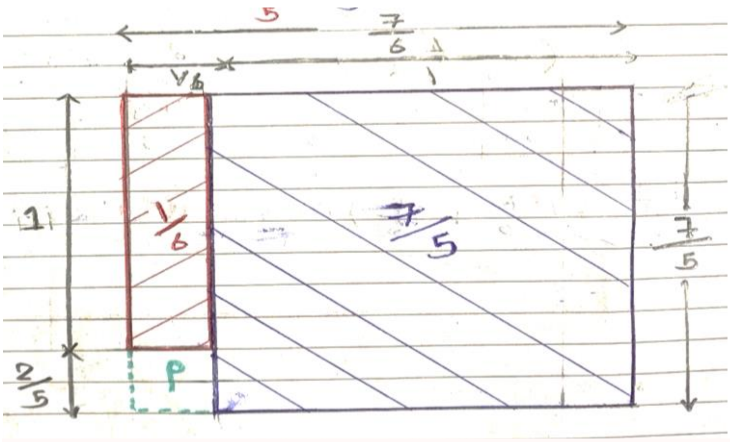
1 and 1/6, (II) and 7/5 and 1 as lengths and widths of two rectangular area models to get the outputs 1/6 and 7/5 (multiplication of each two dimensions), respectively. Hence, to calculate 1/6 + 7/5, the green rectangular area of (7/5 - 1) length and 1/6 width was subtracted from the large rectangle of 7/5 length and (1+ 1/6) width, then the result would be (7/5 x 7/6) - (2/5 x 1/6) = 49/30 - 2/30 = 47/30. In this case, both added fractions and the results were combined into one area model (manner 2 of representations) in which the area of the large rectangle (the result) equals the areas of the rectangles inside (the added fractions).

Although the proposed activity seemed challenging compared to other cases, it might be problematic to facilitate teaching the addition of fractions to young students through it. The reason is that the activity required an understanding of other related concepts, such as area and subtraction, which are usually taught after mastering the addition process.



1/6 +7/5 = (7/6 * 7/5) – (1/6 * 2/5) =

49/30 – 2/30 = 47/30

Case 19

Figure 8. Sample H responses for representing the addition process through the Operator construct

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CONCLUSION

Upon the results detailed above, mathematics teachers' knowledge for teaching the addition of fractions included utilizing the *Part-whole* (the most common), Measure, and *Operator* constructs through three manners of representations. These are (1) *distinct models of the added fractions and the result*, (2) *one model of both the added fractions and the result*, and (3) *models of added fractions only*. Furthermore, several related concepts emerged upon the construct employed; essentially, unit and proportional equivalence and the relationship between the whole and its parts appeared when teachers taught the addition process through the Part-whole construct. Also, improper fractions, fractional units, and the relationship between addition and subtraction were expressed when teachers utilized the Measure construct; besides, the concept of the area of a rectangle as a multiplication process emerged while performing the Operator construct. Therefore, the study recommended professional development courses for mathematics teachers to reinforce these concepts and clarify their effectiveness when teaching fractions and operations on fractions to students. Pharrell to this, future studies might focus on a specific concept (e.g., equivalence, fractional unit, the whole) to be investigated, through which results of the current study could be complemented.

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